JOURNAL OF APPROXIMATION THEORY 3, 164–182 (1970)

# Interpolating Subspaces in Approximation Theory

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*Communicated by E. W. Cheney*

Received July 5, 1969

## 1. INTRODUCTION

There is a very beautiful, now classical, theory associated with the problem of best approximation in $C[a, b]$ by elements of an $n$-dimensional Haar subspace. In particular (cf., e.g., [4]), best approximations are always unique and are characterized by an alternation property; a de la Vallée Poussin theorem provides lower bounds on the error of approximation; best approximations are strongly unique (in the sense of Newman and Shapiro [14]); the metric projection, or best approximation operator, is pointwise Lipschitz continuous; and the so-called "first and second algorithms" of Remez provide effective means for the actual computation of best approximations.

It is natural to ask whether one can extend the notion of a Haar subspace so as to be valid in an *arbitrary* normed linear space, and at the same time preserve as much of the $C[a, b]$ theory as possible. In this paper we introduce the notion of an *interpolating subspace* of a normed linear space. In the particular space $C(T)$, $T$ compact Hausdorff, the interpolating subspaces turn out to be precisely the Haar subspaces. (Recall that an $n$-dimensional subspace $M$ of $C(T)$ is called a *Haar subspace* if every function in $M \sim \{0\}$ has at most $n - 1$ zeros in $T$.) We shall verify that *corresponding to each one of the classical results mentioned in the preceding paragraph for Haar subspaces in $C[a, b]$, there is a strictly analogous result valid for interpolating subspaces in an arbitrary normed linear space.*

Because the richness of the classical Haar subspace theory carries over in toto to the more general case of interpolating subspaces, it might be suspected that interpolating subspaces are rather rare in general normed linear spaces. Indeed, we show (Theorem 3.1) that interpolating subspaces do not exist in those spaces having strictly convex dual spaces. On the other hand, we show (Theorem 3.2) that in $C_0(T)$, $T$ locally compact Hausdorff,

* These authors were supported by grants from the National Science Foundation.

the interpolating subspaces are precisely the Haar subspaces again. (The definition of a Haar subspace in $C_0(T)$ is the same as given for $C(T)$, above.) Also, if $(T, \Sigma, \mu)$ is a $\sigma$-finite measure space, then (Theorem 3.3) $L_1(T, \Sigma, \mu)$ contains an interpolating subspace of dimension $n > 1$ if, and only if, $T$ is the union of at least $n$ atoms. Further, $L_1(T, \Sigma, \mu)$ contains a one-dimensional interpolating subspace if, and only if, $T$ contains an atom. In particular (Corollary 3.4), the space $l_1$ has interpolating subspaces of every finite dimension.

## 2. Definitions, Notation, and Two Basic Results

Let $X$ be a real normed linear space and $X^*$ its dual space. We denote the norm-closed unit balls in each of these spaces by $S(X)$ and $S(X^*)$, respectively. If $K$ is any subset of $X$, ext $K$ denotes the set of extreme points of $K$. If $x_1, ..., x_n$ are linearly independent vectors in $X$, then $[x_1, ..., x_n]$ denotes the $n$-dimensional linear subspace of $X$ generated by these vectors. By subspace we always mean a linear subspace. If $K$ is a subset of $X$ and $x \in X$, an element $x_0 \in K$ is called a *best approximation* to $x$ from $K$ if

$$\| x - x_0 \| = \inf\{\| x - y \| : y \in K\} \equiv d(x, K).$$

If each $x \in X$ has a unique best approximation from $K$, then $K$ is called a *Tchebycheff set*. If $M$ is a subspace of $X$, then

$$M^\perp \equiv \{x^* \in X^* : x^*(y) = 0 \qquad \text{for every } y \in M\}.$$

All other notation or terminology is defined in [7].

Definition. An $n$-dimensional subspace $M$ of $X$ is called an *interpolating subspace* if, for each set of $n$ linearly independent functionals $x_1^*, ..., x_n^*$ in ext $S(X^*)$ and each set of $n$ real scalars $c_1, ..., c_n$, there is a unique element $y \in M$ such that $x_i^*(y) = c_i$ for $i = 1, ..., n$.

Theorem 2.1. *Let $M = [x_1, ..., x_n]$ be an n-dimensional subspace of X. The following statements are equivalent.*

(1) *$M$ is an interpolating subspace.*

(2) *For each set of n linearly independent functionals $x_1^*, ..., x_n^*$ in* ext $S(X^*)$, $\det[x_i^*(x_j)] \neq 0$.

(3) If $x_1^*, ..., x_n^*$ *are n linearly independent functionals in* ext $S(X^*)$, *$y \in M$, and $x_i^*(y) = 0$ for $i = 1, ..., n$, then $y = 0$.*

(4) $M^\perp \cap (\bigcup\{[x_1^*, ..., x_n^*] : x_1^*, ..., x_n^*$ *are linearly independent and lie in* ext $S(X^*)\}) = \{0\}$.

(5) $M^{\perp} \cap [x_1^*,..., x_n^*] = \{0\}$ *for every set of* $n$ *linearly independent functionals* $x_1^*,..., x_n^*$ *in* $\operatorname{ext} S(X^*)$.

(6) $X^* = M^{\perp} \oplus [x_1^*,..., x_n^*]$ *for every set of* $n$ *linearly independent functionals* $x_1^*,..., x_n^*$ *in* $\operatorname{ext} S(X^*)$.

The proof of this theorem is a straightforward application of the definition of an interpolating subspace and is, therefore, omitted.

THEOREM 2.2. *Every interpolating subspace is a Tchebycheff subspace.*

The proof is a simple modification of standard arguments ([16], [6]) and is omitted. Corollary 3.1 below shows that the converse is false.

## 3. EXISTENCE OF INTERPOLATING SUBSPACES IN CONCRETE SPACES

We begin this section by first establishing a "nonexistence" theorem. (Recall that a normed linear space $X$ is called *strictly convex* if $\operatorname{ext} S(X) = \{x \in X : \| x \| = 1\}$.)

THEOREM 3.1. *If* $X$ *is a normed linear space whose dual* $X^*$ *is strictly convex, then* $X$ *has no proper interpolating subspace.*

*Proof.* Clearly, we may assume $\dim X > 1$. Fix an arbitrary integer $n$, $1 \leqslant n < \dim X$, and let $M$ be an $n$-dimensional subspace of $X$. Since $M \neq X$, $M^{\perp}$ must contain a nonzero element $x^*$ by the Hahn-Banach theorem. By the strict convexity of $X^*$, $y^* \equiv (x^*/\| x^* \|) \in \operatorname{ext} S(X^*)$. In particular, $M^{\perp} \cap \operatorname{ext} S(X^*) \supset \{y^*\}$ and, a fortiori,

$$M^{\perp} \cap (\bigcup \{[x_1^*,..., x_n^*] : x_1^*,..., x_n^* \text{ are linearly independent and lie in } \operatorname{ext} S(X^*)\}) \supset \{y^*\}.$$

By Theorem 2.1, $M$ is not an interpolating subspace and the proof is complete.

COROLLARY 3.1. *In an inner product space or any* $L_p(T, \Sigma, \mu)$ *space,* $1 < p < \infty$, *there are no proper interpolating subspaces.*

*Remark.* If $X$ is $n$-dimensional and $M = X$, then $M$ is trivially an interpolating subspace. Indeed, $\det[x_i^*(x_j)] \neq 0$ for *any* set of $n$ linearly independent functionals $x_1^*,..., x_n^*$ in $X^*$ and any basis $x_1,..., x_n$ of $X$ (cf., e.g., [5], p. 26).

If $T$ is a locally compact Hausdorff space, let $C_0(T)$ denote the space of all real-valued continuous functions on $T$ which vanish at infinity, with the supremum norm. Thus, $x \in C_0(T)$ if, and only if, $x$ is continuous and for each $\epsilon > 0$, the set $\{t \in T: |x(t)| \geqslant \epsilon\}$ is compact. In particular, if $T$ is compact, $C_0(T) = C(T)$, the space of *all* real-valued continuous functions on $T$.

THEOREM 3.2. *Let $M$ be a finite-dimensional subspace of $C_0(T)$. The following statements are equivalent.*

(1) *$M$ is an interpolating subspace.*

(2) *$M$ is a Tchebycheff subspace.*

(3) *$M$ is a Haar subspace.*

*Proof.* The equivalence of (1) and (3) follows from part (3) of Theorem 2.1 and the known result that the extreme points of $S[C_0(T)^*]$ are (plus or minus) the point evaluation functionals. The equivalence of (2) and (3) is due to Phelps ([15], p. 250).

COROLLARY 3.2. *A subspace of $c_0$ is a Tchebycheff subspace if, and only if, it is an interpolating subspace.*

This corollary follows by first recalling [10] that $c_0$ *has no infinite-dimensional Tchebycheff subspaces*, and then applying Theorem 3.2.

Let $(T, \Sigma, \mu)$ be a $\sigma$-finite measure space. An *atom* is a set $A \in \Sigma$ with $0 < \mu(A) < \infty$, such that $B \in \Sigma$, $B \subset A$ implies that either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. It is well-known (and easy to prove) that $T$ can have at most countably many atoms. The measure space$(T, \Sigma, \mu)$ is called *nonatomic* if $T$ has no atoms, and it is called *purely atomic* if $T$ is the union of atoms. R. R. Phelps and Henry Dye [15] have shown that *if $T$ has no atoms then $L_1(T, \Sigma, \mu)$ has no finite-dimensional Tchebycheff subspaces* (and, a fortiori, no interpolating subspaces). Sharpening this result, Garkavi [9] established that *$L_1(T, \Sigma, \mu)$ has an n-dimensional Tchebycheff subspace if, and only if, $T$ contains at least n atoms.*

The main result on the existence of interpolating subspaces in $L_1(T, \Sigma, \mu)$ is the following

THEOREM 3.3. *The space $L_1(T, \Sigma, \mu)$ contains an interpolating subspace of dimension $n > 1$ if, and only if, $T$ is the union of at least n atoms. Also, $L_1(T, \Sigma, \mu)$ contains a one-dimensional interpolating subspace if, and only if, $T$ contains an atom.*

As immediate consequences of this theorem, we obtain the following two corollaries.

COROLLARY 3.3. *If the space $L_1(T, \Sigma, \mu)$ contains an interpolating subspace of dimension $> 1$, then $T$ is purely atomic.*

COROLLARY 3.4. *The space $l_1$ has interpolating subspaces of every (finite) dimension.*

We remark that if $(T, \Sigma, \mu)$ is $\sigma$-finite, the condition that $T$ be the union of atoms is equivalent to the condition that $L_1(T, \Sigma, \mu)$ be isometrically isomorphic to $l_1$ or $l_1^m$, depending on whether $T$ is a countable union of atoms or a finite union of $m$ atoms, respectively.

In contrast to Theorem 3.2, *not* every finite-dimensional Tchebycheff subspace in $l_1$ is an interpolating subspace. Indeed, let $X = l_1$ and $M = [e_1, ..., e_n]$, where $e_i$ is the $i$-th unit vector: $e_i = (\delta_{1i}, \delta_{2i}, ...)$. It is easy to verify that $M$ is a Tchebycheff subspace. In fact, if $x = (\xi_1, \xi_2, ...) \in l_1$, then its unique best approximation in $M$ is given by $(\xi_1, ..., \xi_n, 0, ...)$. We identify $l_1^*$ with $l_\infty$ in the usual way. Then each functional $x^* \in \operatorname{ext} S(l_1^*)$ is of the form $x^* = (\sigma_1, \sigma_2, ...)$, where $\sigma_i = \pm 1$ for each $i$. Let $x_1^*, ..., x_n^*$ be any $n$ linearly independent elements of $\operatorname{ext} S(l_1^*)$, each of whose first $n$ coordinates is $+1$. Then $x_i^*(e_j) = 1$ for $i, j = 1, ..., n$, so that $\det[x_i^*(e_j)] = 0$. By Theorem 2.1, $M$ is not as interpolating subspace.

We shall postpone the (rather involved) verification of Theorem 3.3 until the last section, where we also include some results helpful in recognizing and constructing interpolating subspaces in $l_1$.

## 4. CHARACTERIZATION OF BEST APPROXIMATIONS

Let $x, x_1, ..., x_n \in X$, $x_1^*, ..., x_{n+1}^* \in X^*$, and define the determinant $\Delta = \Delta(x, x_1, ..., x_n; x_1^*, ..., x_{n+1}^*)$ by

$$\Delta = \begin{vmatrix} x_1^*(x) & \cdots & x_{n+1}^*(x) \\ x_1^*(x_1) & \cdots & x_{n+1}^*(x_1) \\ \cdots & & \\ x_1^*(x_n) & \cdots & x_{n+1}^*(x_n) \end{vmatrix}. \tag{4.1}$$

The cofactor of $x_i^*(x)$ in $\Delta$ will be denoted by $\Delta_i \equiv \Delta_i(x_1, ..., x_n; x_1^*, ..., x_{n+1}^*)$, i.e.,

$$\Delta_i = (-1)^{i+1} \begin{vmatrix} x_1^*(x_1) & \cdots & x_{i-1}^*(x_1)\, x_{i+1}^*(x_1) & \cdots & x_{n+1}^*(x_1) \\ \cdots & & & & \\ x_1^*(x_n) & \cdots & x_{i-1}^*(x_n)\, x_{i+1}^*(x_n) & \cdots & x_{n+1}^*(x_n) \end{vmatrix}. \tag{4.2}$$

It is worth emphasizing that the cofactors $\Delta_i$ do *not* depend on $x$.

LEMMA 4.1. *Assume $M = [x_1, ..., x_n]$ is an $n$-dimensional interpolating* *subspace in* $X$, $x_1^*,\ldots, x_m^*$ *are* $m \leqslant n+1$ *independent functionals in* ext $S(X^*)$, *and* $\alpha_1,\ldots, \alpha_m$ *are nonzero scalars. Then* $\sum_1^m \alpha_i x_i^* \in M^\perp$ *if, and only if,*

(i) $m = n+1$, *and*

(ii) $\alpha_i = \alpha_{n+1}\Delta_i/\Delta_{n+1}$ $(i = 1,\ldots, n+1)$, *where* $\Delta_i$ *are given by* (4.2).

*In particular,* $\sum_1^{n+1} \Delta_i x_i^* \in M^\perp$.

*Proof.* If $m < n+1$, choose $y \in M$ so that $x_i^*(y) = \alpha_i$ $(i = 1,\ldots, m)$. Then

$$0 = \sum_1^m \alpha_i x_i^*(y) = \sum_1^m \alpha_i^2,$$

which is absurd. Part (ii) follows by using Cramer's rule to solve for $\alpha_1,\ldots, \alpha_n$. The converse follows by observing that $\sum_{i=1}^{n+1} \Delta_i x_i^*(x_j)$ is just the expansion of the determinant $\Delta$ with $x$ replaced by $x_j$, and is, therefore, zero.

The following "alternation" theorem characterizes best approximations from interpolating subspaces.

THEOREM 4.1. *Let* $M = [x_1,\ldots, x_n]$ *be an* $n$*-dimensional interpolating subspace in* $X$, *let* $x \in X \sim M$, *and let* $x_0 \in M$. *Then the following statements are equivalent.*

(1) $x_0$ *is a best approximation to* $x$ *from* $M$.

(2) *There exist* $n+1$ *linearly independent functionals* $x_1^*,\ldots, x_{n+1}^*$ *in* ext $S(X^*)$ *such that*

(a) $x_i^*(x - x_0) = \| x - x_0 \|$ $(i = 1,\ldots, n+1)$,

(b) *The determinants* $\Delta_i$, *defined by Eq.* (4.2), *all have the same sign.*

(3) *There exist* $n+1$ *linearly independent functionals* $x_1^*,\ldots, x_{n+1}^*$ *in* ext $S(X^*)$ *such that*

(a) $x_i^*(x - x_0) = \| x - x_0 \|$ $(i = 1,\ldots, n+1)$,

(b) $\operatorname{sgn}(\Delta_i \Delta) = 1$ $(i = 1,\ldots, n+1)$, *where* $\Delta$ *and* $\Delta_i$ *are as defined in Eqs.* (4.1) *and* (4.2).

(4) *There exist* $n+1$ *linearly independent functionals* $x_1^*,\ldots, x_{n+1}^*$ *in* ext $S(X^*)$ *and* $n+1$ *nonzero scalars* $\alpha_1,\ldots, \alpha_{n+1}$ *such that*

(a) $| x_i^*(x - x_0)| = \| x - x_0 \|$ $(i = 1,\ldots, n+1)$,

(b) $\sum_1^{n+1} \alpha_i x_i^* \in M^\perp$,

(c) $\operatorname{sgn}[\alpha_1 x_1^*(x - x_0)] = \cdots = \operatorname{sgn}[\alpha_{n+1} x_{n+1}^*(x - x_0)]$.

(5) *The zero* $n$ *tuple* $(0,\ldots, 0)$ *is in the convex hull of the set of* $n$ *tuples*

$$\{(x^*(x_1),\ldots, x^*(x_n)) : x^* \in \text{ext}\, S(X^*), \quad x^*(x - x_0) = \| x - x_0 \|\}.$$

The proof is, again, a modification of standard arguments, using lemma 4.1. In particular, in proving the equivalence of (1) and (5), one uses the main characterization theorem of [16].

For our first application of Theorem 4.1, we consider the space $X = C_0(T)$, $T$ locally compact. We can readily deduce:

THEOREM 4.2. *Let $M = [x_1, \ldots, x_n]$ be an n-dimensional interpolating subspace in $C_0(T)$, let $x \in C_0(T) \sim M$, and let $x_0 \in M$. Then $x_0$ is a best approximation to x from M if, and only if, there exist $n + 1$ distinct points $t_1, \ldots, t_{n+1} \in T$ such that*

$$x(t_i) - x_0(t_i) = \operatorname{sgn}(D_i D) \| x - x_0 \| \qquad (i = 1, \ldots, n + 1), \tag{a}$$

*where*

$$D \equiv \begin{vmatrix} x(t_1) & \cdots & x(t_{n+1}) \\ x_1(t_1) & \cdots & x_1(t_{n+1}) \\ \cdots & & \\ x_n(t_1) & \cdots & x_n(t_{n+1}) \end{vmatrix} \neq 0 \tag{b}$$

*and $D_i$ is the cofactor of $x(t_i)$ in D.*

Theorem 4.2 was, in essence, established by Bram [2], who gave a direct proof. In the particular case when $T$ is compact, Theorem 4.2 was proved by Zuhovitki [18]. If we further specialize and take $T$ to be an *interval on the real line*, we obtain the classical alternation theorem.

As another application of Theorem 4.1, we consider the space $L_1 \equiv L_1(T, \Sigma, \mu)$, where $T$ is the union of (at most) countably many atoms, say $T = \bigcup_{i \in I} A_i$. Since each measurable function $x$ must be constant almost everywhere on an atom and since $L_1^* = L_\infty$, it is easy to verify that each $x^* \in \operatorname{ext} S(L_1^*)$ has the representation

$$x^*(x) = \sum_{i \in I} x(A_i)\, \sigma(A_i)\, \mu(A_i), \qquad x \in L_1,$$

where $|\sigma(A_i)| = 1$ and where $x(A_i)$ denotes the constant value which $x$ has a.e. on $A_i$. For any $x \in L_1$, we denote the set $\{i \in I : x(A_i) = 0\}$ by $Z(x)$. If $S$ is a set, then card $S$ will denote the cardinality of $S$.

THEOREM 4.3. *Let $M = [x_1, \ldots, x_n]$ be an n-dimensional interpolating subspace in $L_1$, let $x \in L_1 \sim M$, and let $x_0 \in M$. The following statements are equivalent.*

(1) *$x_0$ is a best approximation to $x$ from $M$.*

(2) *There exist $n+1$ linearly independent measurable functions $\sigma_1,\ldots,\sigma_{n+1}$, with $|\sigma_i| = 1$ $(i = 1,\ldots, n+1)$ such that*

(a) $\sigma_1(A_i) = \cdots = \sigma_{n+1}(A_i) = \operatorname{sgn}[x(A_i) - x_0(A_i)]$ for each $i \in I \sim Z(x - x_0)$,

(b) card $Z(x - x_0) \geqslant n$,

(c) *The $n$ tuple*

$$\left(\sum_{i\in I} \operatorname{sgn}[x(A_i) - x_0(A_i)]\, x_1(A_i)\, \mu(A_i),\ldots, \sum_{i\in I} \operatorname{sgn}[x(A_i) - x_0(A_i)]\, x_n(A_i)\, \mu(A_i)\right)$$

*is in the convex hull of the set of $n$ tuples*

$$\left\{\left(\sum_{i\in Z(x-x_0)} \sigma_j(A_i)\, x_1(A_i)\, \mu(A_i),\ldots, \sum_{i\in Z(x-x_0)} \sigma_j(A_i)\, x_n(A_i)\, \mu(A_i)\right) : j = 1,\ldots, n+1\right\}.$$

(3) Card $Z(x - x_0) \geqslant n$ *and*

$$\left|\sum_{i\in I} \operatorname{sgn}[x(A_i) - x_0(A_i)]\, y(A_i)\, \mu(A_i)\right| \leqslant \sum_{i\in Z(x-x_0)} |y(A_i)|\, \mu(A_i) \qquad (4.3)$$

*for every $y \in M$.*

(4) *Inequality* (4.3) *is valid for every $y \in M$.*

We omit the straightforward proof, observing only that the implication (4) $\Rightarrow$ (1) follows by an application of a result of H. S. Shapiro (cf., e.g., [11, Corollary 1.4]).

The space $l_1$ is the most important example of the type we have been considering. (In fact, $l_1 = L_1(T, \Sigma, \mu)$, where $T = \{1, 2, 3,\ldots\}$, $\Sigma$ is the collection of all subsets of $T$, and $\mu$ is the "counting" measure: $\mu(B) = \operatorname{card}(B)$.) Thus, we immediately deduce from Theorem 4.3 the following

Corollary 4.1. *Let $M = [x_1,\ldots, x_n]$ be an $n$-dimensional interpolating subspace in $l_1$, let $x = (\xi_1, \xi_2,\ldots) \in l_1 \sim M$, let $x_0 \in M$, and set $x_i = (\xi_{i1}, \xi_{i2},\ldots)$ $(i = 0, 1,\ldots, n)$. The following statements are equivalent.*

(1) *$x_0$ is a best approximation to $x$ from $M$.*

(2) *There exist $n+1$ linearly independent vectors $\sigma_i = (\sigma_{i1}, \sigma_{i2},\ldots) \in l_\infty$, with $|\sigma_{ij}| = 1$, such that*

(a) *For each* $i = 1,\dots, n+1$, $\sigma_{ij} = \operatorname{sgn}[\xi_j - \xi_{0j}]$ *whenever* $\xi_j \neq \xi_{0j}$,
(b) Card $Z(x - x_0) \geqslant n$ $[Z(x - x_0) \equiv \{k : \xi_k = \xi_{0k}\}]$,
(c) *The n tuple*

$$\left(\sum_1^\infty \operatorname{sgn}(\xi_i - \xi_{0i})\, \xi_{1i}, \dots, \sum_1^\infty \operatorname{sgn}(\xi_i - \xi_{0i})\, \xi_{ni}\right)$$

*is in the convex hull of the set of n tuples*

$$\left\{\left(\sum_{i \in Z(x-x_0)} \sigma_{ji}\xi_{1i}, \dots, \sum_{i \in Z(x-x_0)} \sigma_{ji}\xi_{ni}\right) : j = 1,\dots, n+1\right\}.$$

(3) Card $Z(x - x_0) \geqslant n$ *and*

$$\left| \sum_1^\infty \operatorname{sgn}(\xi_i - \xi_{0i})\, \eta_i \right| \leqslant \sum_{i \in Z(x-x_0)} |\eta_i| \tag{4.7}$$

*for every* $y = (\eta_1, \eta_2, \dots) \in M$.

(4) *Inequality* (4.7) *is valid for every* $y = (\eta_1, \eta_2, \dots) \in M$.

## 5. Error of Approximation

The first result of this section provides a useful relation for obtaining the error of approximation of a vector by elements of an interpolating subspace, and in particular, for obtaining lower bounds on this approximation error.

Theorem 5.1. *Let M be an n-dimensional interpolating subspace in X and let* $x \in X$. *Then*

$$d(x, M) = \max \left| \sum_1^{n+1} \lambda_i x_i^*(x) \right|,$$

*where the maximum is taken over all sets of* $n+1$ *linearly independent functionals* $x_1^*,\dots, x_{n+1}^*$ *in* ext $S(X^*)$, *and* $\lambda_i \equiv \lambda_i(x_1^*,\dots, x_{n+1}^*) = \Delta_i / \sum_1^{n+1} \Delta_k$, *where the determinants* $\Delta_i \equiv \Delta_i(x_1^*,\dots, x_{n+1}^*)$ *are defined by Eq.* (4.2). *They all have the same sign.*

*Proof.* It is a well-known consequence of the Hahn-Banach theorem that (for an arbitrary subspace $M$)

$$d(x, M) = \max\{|x^*(x)| : x^* \in S(X^*) \cap M^\perp\}.$$

Moreover, when $M$ is $n$-dimensional, we may restrict the search for a maximum to those $x^*$ of the form $x^* = \sum_1^m \lambda_i x_i^*$, where $x_i^* \in \operatorname{ext} S(X^*)$, $\lambda_i > 0$, $\sum_1^m \lambda_i = 1$, and $m \leqslant n + 1$ (cf., e.g., [17]). Our conclusion now follows immediately from Lemma 4.1.

With the help of Theorem 5.1, we can deduce the following generalized "de la Vallée Poussin theorem," from which the classical result under that name follows easily.

THEOREM 5.2. *Let $M$ be an $n$-dimensional interpolating subspace of $X$ and let $x \in X$. Suppose there exist a $y \in M$ and $n + 1$ linearly independent functionals $x_1^*,..., x_{n+1}^*$ in* ext $S(X^*)$ *such that*

$$[\Delta_i x_i^*(x - y)][\Delta_{i+1} x_{i+1}^*(x - y)] > 0 \qquad (i = 1,..., n),$$

*where the determinants $\Delta_i$ are defined by Eq.* (4.2). *Then*

$$\min_i | x_i^*(x - y)| \leqslant d(x, M).$$

*Also, if equality holds, then $| x_i^*(x - y)| = d(x, M)$ for every $i$.*

## 6. Continuity of Best Approximations

We now state a "strong uniqueness" theorem which generalizes a result of Newman and Shapiro [14]. If $M$ is an interpolating subspace in $X$, we denote the unique best approximation from $M$ to any $x \in X$ by $B_M(x)$. The operator $B_M$ is called the *metric projection* onto $M$.

THEOREM 6.1. *Let $M$ be an interpolating subspace in $X$. Then, for each $x \in X$, there exists a constant $\gamma = \gamma(x)$ with $0 < \gamma \leqslant 1$, such that*

$$\| x - y \| \geqslant \| x - B_M(x)\| + \gamma \| B_M(x) - y \|$$

*for every $y \in M$.*

Cheney and Wulbert (unpublished, 1967) have obtained a result slightly stronger than Theorem 6.1. Their proof, as well as ours, is an obvious modification of the proof of the Newman-Shapiro Theorem as given by Cheney [3], [4].

Freud [8], in essence, showed that the metric projection onto a Haar subspace in $C[a, b]$ is pointwise Lipschitz continuous. Cheney [4], p. 82, observed that this fact is a consequence only of the strong uniqueness theorem, so that it is equally valid for our situation. Thus, we immediately obtain from Theorem 6.1, THEOREM 6.2. *Let $M$ be an interpolating subspace of $X$. Then for each $x \in X$ there exists a constant $\lambda = \lambda(x) > 0$ such that*

$$\| B_M(x) - B_M(z)\| \leqslant \lambda \| x - z \|$$

*for every $z \in X$.*

## 7. ALGORITHMS FOR CONSTRUCTING BEST APPROXIMATIONS

We shall consider two algorithms for the construction of best approximations from interpolating subspaces.

Let $M$ be an $n$-dimensional subspace of $X$, let $x \in X \sim M$, and let $\Gamma$ be any set of functionals in $X^*$ of norm 1 such that for each $z \in M \oplus [x]$, there is an $x^* \in \Gamma$ with $x^*(z) = \| z \|$. In [1], Akilov and Rubinov have described an algorithm—a generalization of the "first algorithm" of Remez—for the construction of a best approximation to $x$ from $M$. If we specialize their result to the case where $M$ is an interpolating subspace and where $\Gamma = \text{ext}\, S(X^*)$, the algorithm may be described as follows.

Let $x_1^*,..., x_n^* \in \Gamma$. For each $m \geqslant n$, select $y_m \in M$ and $x_{n+1}^* \in \Gamma$ so that

$$\max_{k \leqslant m} | x_k^*(x - y_m)| = \min_{y \in M} \max_{k \leqslant m} | x_k^*(x - y)|$$

and

$$| x_{m+1}^*(x - y_m)| = \| x - y_m \|.$$

Introducing the notation $e_m = \| x - y_m \|$, $\| z \|_m = \max_{k \leqslant m} | x_k^*(z)|$, and $\lambda_m = \| x - y_m \|_m$, the effectiveness of this algorithm can be summarized in the following

THEOREM. (i) $\lambda_n \leqslant \lambda_{n+1} \leqslant \cdots \leqslant d(x, M) \leqslant e_m$ *for all* $m \geqslant n$, *and* $\lim \lambda_m = d(x, M) = \lim e_m$.

(ii) *The sequence $\{y_m\}$ converges to the unique best approximation of $x$ from $M$.*

Laurent [12] has recently given a generalization of the "second algorithm" of Remez. It is valid for $n$-dimensional subspaces $M = [x_1 ,..., x_n]$ of a normed linear space $X$ which satisfy the condition:

(L) For each set of $n$ linearly independent functionals $x_1^* ..., x_n^*$ in the weak* closure of ext $S(X^*)$, $\det[x_i^*(x_j)] \neq 0$.

In particular, any subspace with property (L) is necessarily an interpolating subspace. In the special cases when $X = C(T)$, $T$ compact Hausdorff, or when $X = L_1(T, \Sigma, \mu)$, where $T$ is (at most) a countable union of atoms, it can be shown that ext $S(X^*)$ is weak* closed. Hence, in *these special cases* property (L) is *equivalent* to the condition that $M$ be an interpolating subspace. However, in the space $c_0$, for example, there are interpolating subspaces of every finite dimension. Since 0 is in the weak* closure of ext $S(c_0{}^*)$, it follows that *no* subspace of $c_0$ has property (L). We do not know whether this algorithm is still valid for interpolating subspaces in $X$ for which ext $S(X^*)$ is not weak* closed.

A detailed description of the algorithm of Laurent would take us too far astray. We mention, only, that it is a convergent scheme.

We conclude this section by observing that if the $n + 1$ functionals $x_1{}^*,\ldots, x_{n+1}^*$ in the characterization Theorem 4.1 are known, then it is possible to determine the best approximation, as well as the error of approximation, by simple solving a *linear* system of $n + 1$ equations. Indeed, suppose $M = [x_1, \ldots, x_n]$ is an $n$-dimensional interpolating subspace in $X$, $x \in X \sim M$, and $x_0 = \sum_1^n \alpha_i x_i$ is the best approximation to $x$ from $M$. Suppose that $x_1{}^*,\ldots, x_{n+1}^*$ are any $n + 1$ linearly independent functionals in ext $S(X^*)$, whose existence is guaranteed by Theorem 4.1. Then, in particular,

$$x_i{}^*(x_0) + d = x_i{}^*(x) \qquad (i = 1,\ldots, n + 1),$$

where $d = d(x, M)$. Substituting for $x_0$ in these equations, we get

$$\sum_{j=1}^{n} \alpha_j x_i{}^*(x_j) + d = x_i{}^*(x) \qquad (i = 1,\ldots, n + 1).$$

This system can now be solved by Cramer's rule to determine the unknowns $\alpha_i$ (and hence, $x_0$) and $d$.

We remark that the Laurent algorithm involves solving a sequence of such $(n + 1)$-st order linear systems.

## 8. Proof of Theorem 3.3

The proof of Theorem 3.3 will be based on a number of preliminary results which we now establish.

Throughout this section, $n$ denotes a fixed positive integer. For each $m \geqslant n$, we consider the linear space $\mathscr{M}_m$ of all real $n \times m$ matrices $E = (e_{ij})$, with the norm $\| E \| = \max_{i,j} | e_{ij} |$. If $E$ is an $n \times k$ matrix, with $k \leqslant m$, we identify $E$ with $\tilde{E}$, where $\tilde{E} \in \mathscr{M}_m$ is the partitioned matrix

$$\tilde{E} = [\underbrace{E}_{k} \ \vdots \ \underbrace{0}_{m-k}]\}n,$$

and 0 is the $n \times (m - k)$ matrix consisting entirely of zeros. In this way, we have $\mathscr{M}_{m_1} \subset \mathscr{M}_{m_2}$ if $m_1 \leqslant m_2$.

LEMMA 8.1. *Assume $m \geqslant n$, $B_0 \in \mathscr{M}_m$, $E$ is an $m \times n$ matrix of rank $n$, and $\epsilon > 0$. Then there is a matrix $B_1 \in \mathscr{M}_m$ such that*

(1) $\| B_1 - B_0 \| < \epsilon$,

(2) $\det B_1 E \neq 0$.

*Proof.* Define a function of $nm$ real variables $x_{ij}$ $(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m)$ by

$$p(x_{11}, ..., x_{nm}) = \det[(B_0 + U) E],$$

where

$$U = \begin{bmatrix} x_{11} & \cdots & x_{1m} \\ \cdots & & \\ x_{n1} & \cdots & x_{nm} \end{bmatrix}.$$

$p$ is a polynomial in the variables $x_{ij}$ which is not identically 0, since there clearly is some matrix $B \in \mathscr{M}_m$ such that $\det BE \neq 0$. It follows that $p$ cannot vanish identically on any neighborhood of the origin. Hence, there are values $x_{ij}$ with $| x_{ij} | < \epsilon$ so that $p(x_{11}, ..., x_{nm}) \neq 0$. Taking $B_1 = B_0 + U$ completes the proof.

LEMMA 8.2. *Assume $m \geqslant n$, $B_0 \in \mathscr{M}_m$, $E_1, ..., E_r$ are $m \times n$ matrices of rank $n$, and $\epsilon > 0$. Then there exist $B_1 \in \mathscr{M}_m$ and $\delta > 0$, with the following properties*:

(1) $\| B_1 - B_0 \| < \epsilon$;

(2) *if $B \in \mathscr{M}_m$, $\| B - B_1 \| < \delta$, and $U$ is an $n \times n$ matrix with $\| U \| < \delta$, then*

$$\det(BE_i + U) \neq 0 \qquad (i = 1, ..., r).$$

*Proof.* We proceed by induction on $r$. Assume $r = 1$. Then by Lemma 8.1, there is a matrix $B_1 \in \mathscr{M}_m$ such that $\| B_1 - B_0 \| < \epsilon$ and $\det B_1 E_1 \neq 0$. Now, $\det(BE_1 + U)$, regarded as a function of $nm + n^2$ variables (as $B$ varies over $\mathscr{M}_m$ and $U$ varies over all $n \times n$ matrices), is continuous. Hence, there exists a $\delta > 0$ such that if $\| B - B_1 \| < \delta$ and $\| U \| < \delta$, then $\det(BE_1 + U) \neq 0$. Now assume $r > 1$ and that a matrix $B_1' \in \mathscr{M}_m$ and a $\delta' > 0$ have been determined so that $\| B_1' - B_0 \| < \epsilon/2$ and such that $\| B - B_1' \| < \delta'$, $\| U \| < \delta'$ imply $\det(BE_i + U) \neq 0$ for $i = 1, ..., r - 1$. From the case $r = 1$, there is a matrix $B_1'' \in \mathscr{M}_m$ and a $\delta_1 > 0$ such that $\| B_1' - B_1'' \| < \min\{\epsilon/2, \delta'/2\}$ and such that $\| B - B_1'' \| < \delta_1$, $\| U \| < \delta_1$

imply $\det(BE_r + U) \neq 0$. The lemma now follows by taking $B_1 = B_1''$ and $\delta = \min\{\delta_1, \delta'/2\}$.

LEMMA 8.3. *There exists an* $n \times \infty$ *matrix* $B = (b_{ij})$ $(i = 1,..., n;$ $j = 1, 2,...)$ *with* $\sum_{j=1}^{\infty} | b_{ij} | < \infty$, *having the property that* $\det BE \neq 0$ *for every* $\infty \times n$ *matrix* $E$ *of rank* $n$ *whose entries are restricted to the values* $\pm 1$.

*Proof.* We shall obtain $B$ as the limit of a sequence of matrices $B_k$ which we now construct. For each $k = 0, 1, 2,...$, let $\mathscr{E}_k$ denote the finite set of all $(n + k) \times n$ matrices of rank $n$ with entries $\pm 1$. By Lemma 8.2, we construct a sequence of matrices $B_0, B_1,...,$ where $B_k = (b_{ij}^{(k)}) \in \mathscr{M}_{n+k}$, and a corresponding sequence of positive numbers $\delta_0, \delta_1,...,$ having the following properties:

(i) $0 < \delta_0 < 1$, $0 < \delta_{k+1} < \delta_k/2$.

(ii) $\| B_k - B_{k+1} \| < \delta_k/4$ $(k = 0, 1,...)$.

(iii) If $B \in \mathscr{M}_{n+k}$, with $\| B - B_k \| < \delta_k$, and if $U$ is any $n \times n$ matrix with $\| U \| < \delta_k$, then $\det(BE_k + U) \neq 0$ for every $E_k \in \mathscr{E}_k$.

In particular, it follows from (ii) that $| b_{i,n+k+1}^{(k+1)} | < \delta_k/4$ for $i = 1,..., n$, and $k = 1, 2,...$. If we identify each $B_k$ with the $n \times \infty$ partitioned matrix

$$\begin{bmatrix} B_k & \vdots & 0 \quad 0 \quad \cdots \\ & \vdots & \cdots \\ & \vdots & 0 \quad 0 \quad \cdots \end{bmatrix},$$

then the sequence $B_k$ converge entrywise to some $n \times \infty$ matrix, $B = (b_{ij})$. To see this, we note that for each $i = 1,..., n$, $j = 1, 2,...,$ and $p > 0$,

$$\begin{aligned} | b_{ij}^{(k)} - b_{ij}^{(k+p)} | &\leqslant \| B_k - B_{k+p} \| \\ &\leqslant \| B_k - B_{k+1} \| + \cdots + \| B_{k+p-1} - B_{k+p} \| \\ &< \delta_k/4 + \cdots + \delta_{k+p-1}/4 < \tfrac{1}{4}(\delta_k + \delta_{k+1} + \cdots) \\ &< \tfrac{1}{2}\delta_k, \end{aligned}$$

and $\delta_k \to 0$ as $k \to \infty$. For each $k = 0, 1,...$, let $B_k'$ be the matrix in $\mathscr{M}_{n+k}$ consisting of the first $n + k$ columns of $B$. By our construction, we have

$$\| B_k' - B_k \| \leqslant \tfrac{1}{2}\delta_k < \delta_k .$$

Also,

$$\begin{aligned} | b_{i,n+k+j} | &\leqslant | b_{i,n+k+j} - b_{i,n+k+j}^{(k+j)} | + | b_{i,n+k+j}^{(k+j)} | \\ &\leqslant \tfrac{1}{2}\delta_{k+j} + \tfrac{1}{4}\delta_{k+j-1} < \tfrac{1}{2}\delta_{k+j-1} < \frac{1}{2^j}\,\delta_k . \end{aligned}$$

Hence, for each $i = 1, \ldots, n$,

$$\sum_{j=1}^{\infty} | b_{i,n+k+j} | < \delta_k .$$

In particular, each row vector of $B$ is in $l_1$.

Now, let $E = (\sigma_{ij})$ be any $\infty \times n$ matrix with rank $n$ and entries $\sigma_{ij} = \pm 1$. Then, for some $k \geqslant 0$, the $(n + k) \times n$ matrix $E_k$ consisting of the first $n + k$ rows of $E$ has rank $n$. By definition, $E_k \in \mathscr{E}_k$. Let $C = BE = (c_{ij})$. Then, for $1 \leqslant i, j \leqslant n$,

$$c_{ij} = \sum_{r=1}^{\infty} b_{ir}\sigma_{rj} = \sum_{r=1}^{n+k} b_{ir}\sigma_{rj} + \sum_{r=n+k+1}^{\infty} b_{ir}\sigma_{rj} .$$

Thus, $BE = B_k'E_k + U$, where $U$ is the $n \times n$ matrix whose $i, j$-th entry is

$$u_{ij} = \sum_{r=n+k+1}^{\infty} b_{ir}\sigma_{rj} .$$

Hence, $| u_{ij} | \leqslant \sum_{r=n+k+1}^{\infty} | b_{ir} | < \delta_k$, for $1 \leqslant i,\ j \leqslant n$, i.e., $\| U \| < \delta_k$. Since $\| B_k' - B_k \| < \delta_k$ and $E_k \in \mathscr{E}_k$, it follows from the construction that

$$\det BE = \det(B_k'E_k + U) \neq 0,$$

and this completes the proof.

We can now easily prove half of Theorem 3.3.

THEOREM 8.1. *Let $(T, \Sigma, \mu)$ be a $\sigma$-finite measure space such that $T$ is the union of at least $n$ atoms. Then $L_1(T, \Sigma, \mu)$ contains an interpolating subspace of dimension $n$.*

*Proof.* We shall assume that $T$ is a countable union of atoms: $T = \bigcup_1^\infty A_i$, where $\mu(A_i \cap A_j) = 0$ if $i \neq j$. The case where $T$ is only a finite union of atoms can be treated in a similar manner. We assume that $A_i \cap A_j = \phi$ if $i \neq j$ (by neglecting certain sets of measure zero). Each functional $x^* \in \operatorname{ext} S(L_1{}^*)$ is of the form

$$x^*(x) = \int_T xg\, d\mu, \qquad x \in L_1 ,$$

for some $g \in L_\infty$ with $| g | = 1$. Hence,

$$x^*(x) = \sum_1^{\infty} x(A_i)\, \sigma_i \mu(A_i),$$

where $x(A_i)$ is the constant value which $x$ has a.e. on $A_i$, and $g = \sigma_i (= \pm 1)$

on $A_i$. Let $B = (b_{ij})$ be the $n \times \infty$ matrix whose existence is guaranteed by Lemma 8.3. Define $n$ functions $x_1, \ldots, x_n$ by

$$x_k(t) = b_{ki}\mu(A_i)^{-1}, \qquad \text{if} \quad t \in A_i \quad (i = 1, 2, \ldots).$$

Then

$$\int_T | x_k | \, d\mu = \sum_{i=1}^{\infty} | b_{ki} | < \infty, \qquad \text{for } k = 1, \ldots, n,$$

i.e., $x_1, \ldots, x_n$ are in $L_1$. Also, $x_1, \ldots, x_n$ are linearly independent. This follows from the fact that $B$ has rank $n$. If $x_1^*, \ldots, x_n^*$ are linearly independent functionals in ext $S(L_1^*)$, then

$$x_i^*(x) = \sum_{j=1}^{\infty} x(A_j)\, \sigma_{ji}\mu(A_j) \qquad (i = 1, \ldots, n),$$

where $\sigma_{ij} = \pm 1$. It follows that the vectors $(\sigma_{1j}, \sigma_{2j}, \ldots)$ are linearly independent (as elements of $l_\infty$). Hence, letting $E$ denote the $\infty \times n$ matrix

$$E = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & & \vdots \end{bmatrix},$$

we see that $E$ has rank $n$, so that, by Lemma 8.3,

$$\det[x_i^*(x_j)] = \det BE \neq 0.$$

Thus, $[x_1, \ldots, x_n]$ is an $n$-dimensional interpolating subspace in $L_1$ and the proof is complete.

By using the remark following Corollary 3.4, the above proof could have been slightly simplified by assuming $L_1(T, \Sigma, \mu) = l_1$ or $l_1^m$.

LEMMA 8.4. *Let $(T, \Sigma, \mu)$ be a σ-finite non-atomic measure space and let $x \in L_1(T, \Sigma, \mu)$. For any positive integer n, there exist n linearly independent functions $y_1, \ldots, y_n$ in* ext $S[L_\infty(T, \Sigma, \mu)]$ *such that*

$$\int_T y_i x \, d\mu = 0, \qquad \text{for } i = 1, \ldots, n.$$

*Proof.* Define a (signed) measure $\nu$ by

$$\nu(S) = \int_S x \, d\mu, \qquad \text{for every } S \in \Sigma.$$

Then $\nu$ is nonatomic, i.e., both $\nu^+$ and $\nu^-$ are nonatomic. An application of

Liapounoff's convexity theorem [13] shows that $T$ may be decomposed into disjoint sets $A$ and $B$ such that

$$\frac{1}{2}\int_T x\,d\mu = \int_A x\,d\mu = \int_B x\,d\mu.$$

By repeated application of the above, if $k$ is a positive integer, $T$ may be decomposed into disjoint sets $T_1, T_2, \ldots, T_{2^k}$ such that

$$\frac{1}{2^k}\int_T x\,d\mu = \int_{T_i} x\,d\mu, \qquad \text{for } i = 1, \ldots, 2^k.$$

Now, if $y$ is a function constantly equal to 1 on exactly half of the $2^k$ sets and constantly $-1$ on the other half, then $y \in \operatorname{ext} S[L_\infty(T, \Sigma, \mu)]$ and

$$\int_T yx\,d\mu = 0.$$

To complete the proof, we observe that it is possible to choose $k$ large enough so that there exists a linearly independent set of $n$ such functions $y$. Indeed, if $2^k \geqslant 2n$ and if we choose $y_i$ $(i = 1, \ldots, n)$ to be 1 on half of the $2^k$ sets and $-1$ on the other half, and so that we also have

$$y_i(t) = \begin{cases} 1 & \text{if } t \in T_1 \cup T_2 \cup \cdots \cup T_i, \\ -1 & \text{if } t \in T_{i+1} \cup \cdots \cup T_n, \end{cases}$$

then these $y_i$ work.

The above result is related to a theorem of Phelps ([15], Theorem 1.8).

*If the σ-finite measure space $(T, \Sigma, \mu)$ contains an atom, then $L_1(T, \Sigma, \mu)$ always contains a one-dimensional interpolating subspace.* For if $A$ is an atom, define $x$ by

$$x(t) = \begin{cases} 1 & \text{if } t \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then $x \in L_1(T, \Sigma, \mu)$ and $x^*(x) = \pm\mu(A) \neq 0$ for each $x^* \in \operatorname{ext} S(L_1{}^*)$. Hence, $[x]$ is a one-dimensional interpolating subspace in $L_1$.

The above remark along with the following theorem establish the second half of Theorem 3.3.

THEOREM 8.2. *Let $(T, \Sigma, \mu)$ be a σ-finite measure space. If $T$ is not a union of atoms, then $L_1(T, \Sigma, \mu)$ has no interpolating subspace of dimension $n > 1$. If $T$ contains no atoms, then $L_1(T, \Sigma, \mu)$ contains no interpolating subspace.*

*Proof.* The second assertion is clear from Lemma 8.4. (It is also a consequence of the theorem of Phelps and Dye mentioned in Section 3.) For the first assertion, we may assume that $T$ contains some atoms. Let $A$ be the union of the atoms in $T$. Let $M$ be a subspace of $L_1$ of dimension $n > 1$. Then there is a nonzero $x \in M$ such that

$$\int_A x \, d\mu = 0.$$

Applying Lemma 8.4 to the measure space $(T \sim A, \Sigma, \mu)$, we get $n$ linearly independent functions $y_1, ..., y_n$ in ext $S[L_\infty(T \sim A, \Sigma, \mu)]$ such that

$$\int_{T \sim A} y_i x \, d\mu = 0 \qquad (i = 1, ..., n).$$

Now define

$$y_i' = \begin{cases} y_i & \text{on } T \sim A, \\ 1 & \text{on } A. \end{cases}$$

Then $y_1', ..., y_n'$ are linearly independent functions in ext $S[L_\infty(T, \Sigma, \mu)]$ and

$$\int_T y_i' x \, d\mu = 0 \qquad (i = 1, ..., n).$$

Defining $x_i^*$ by

$$x_i^*(z) = \int_T z y_i' \, d\mu \qquad \text{for all } z \in L_1,$$

we see that $x_1^*, ..., x_n^*$ are linearly independent functionals in ext $S(L_1^*)$ and $x_i^*(x) = 0$ $(i = 1, ..., n)$. Thus, $M$ is not an interpolating subspace and the proof is complete.

It would be of some practical use to have a *constructive* proof of Theorem 8.1. Along these lines we make the following conjecture:

The vectors $x_i = (1, r_i, r_i^2, r_i^3, ...) \in l_1$ $(i = 1, ..., n, n > 1)$ span an $n$-dimensional interpolating subspace, if $0 < r_1 < r_2 < \cdots < r_n < \frac{1}{2}$ and the ratio $r_j/r_{j+1}$ is "sufficiently" small $(j = 1, ..., n - 1)$.

We have thus far verified this conjecture for $n \leqslant 4$. In the absence of a complete proof of the conjecture, the following results, which can be easily verified, might be useful in recognizing interpolating subspaces in $l_1$.

PROPOSITION 8.1. *Let $M = [x_1, ..., x_n]$, $n \geqslant 2$, be an $n$-dimensional interpolating subspace in $l_1$. Then for no $j$ is it possible that the $j$-th coordinates of $x_1, x_2, ..., x_n$ are all zero.*

PROPOSITION 8.2. *Let $M$ be an interpolating subspace of dimension $n > 2$ in $l_1$. Then for every pair of linearly independent vectors in $M$, the number of $j$'s, such that the $j$-th coordinate of both vectors is 0, is $\leqslant n - 2$.*

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